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Numerical Quadrature of Improper Integrals and the Dominated Integral

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1. INTRODUCTION, STATEMENT OF THEOREMS, AND EXAMPLES

The main purpose of this article is to show that the dominated integral introduced in [9] is a natural and powerful tool in the study of the application of quadrature formulas to the numerical evaluation of improper Riemann integrals. More accurately, we show that a function f on $(0, 1]$ is dominantly integrable if and only if every sequence of quadrature formulas, of some reasonable, natural form, when applied to f, converges to (the improper Riemann integral) $\int_{0+}^{1} f(t)\, dt$. The types of sequences of quadrature formulas which we shall treat in examples include all sequences of compound rules on $[0, 1]$ not involving $f(0)$, and integrating 1 exactly, and sequences of n-point Gauss-type quadrature formulas, for $n = 1, 2,\ldots$. This article was stimulated by the work of Davis and Rabinowitz [1], who showed that quadrature formulas whose use can be justified for the evaluation of proper Riemann integrals can be sometimes also used for the evaluation of improper Riemann integrals. Work in this area was continued by other authors (cf. [3, 4, 8, 10]).

By Theorem 3 of [9] a number of properties are equivalent to dominant integrability. Perhaps the simplest of these equivalences is that f is dominantly integrable if and only if f is Riemann integrable on each $[a, b] \subset (0, 1]$, and there exists a function h, monotone nonincreasing and improperly Riemann integrable on $(0, 1]$ such that $h(t) \geqslant |f(t)|$ throughout $(0, 1]$. Theorem 1 below gives yet another property equivalent to dominant integrability, one which is useful in the numerical evaluation of $\int_{0+}^{1} f(t)\, dt$, the improper Riemann integral of f on $(0, 1]$.

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Let $0 < \delta < 1$, and let n be a positive integer. Let f be a complex function on $(0, 1]$. We shall often consider sums

$$\sum_{j=1}^{n} f(\tau_j)(t_j - t_{j-1}),$$

where

$$0 = t_0 < t_1 < \cdots < t_n = 1;$$

and

$$\max(t_{j-1}, \delta t_j) \leqslant \tau_j \leqslant t_j, \qquad j = 1, 2,..., n. \tag{1}$$

THEOREM 1. *A complex function f on* $(0, 1]$ *is dominantly integrable if and only if there exists a complex number I such that for each δ, $0 < \delta < 1$ (or for some fixed δ_1, $0 < \delta_1 < 1$) and each $\epsilon > 0$ there exists an $m = m(\delta, \epsilon) > 0$ (an $m = m(\epsilon) > 0$) such that*

$$\left| I - \sum_{j=1}^{n} f(\tau_j)(t_j - t_{j-1}) \right| < \epsilon \tag{2}$$

whenever (1) *holds (whenever* (1) *holds with δ_1 replacing δ) and each $t_j - t_{j-1} < m$. Further, such an I is necessarily $\int_{0+}^{1} f(t)\,dt$.*

EXAMPLE 1. Let f be a complex function on $(0, 1]$, and let $0 < \delta < 1$. Let $(R_n(f))_{n=1}^{\infty}$ be a sequence of Riemann sums:

$$R_n(f) \equiv \sum_{j=1}^{n} f(\tau_j^{(n)})(t_j^{(n)} - t_{j-1}^{(n)})$$

where $0 = t_0^{(n)} < t_1^{(n)} < \cdots < t_n^{(n)} = 1$, for $n = 1, 2,...$;

$$\lim_{n\to\infty} \max_{1\leqslant j\leqslant n} (t_j^{(n)} - t_{j-1}^{(n)}) = 0; \qquad \text{and} \qquad \max(t_{j-1}^{(n)}, \delta t_j^{(n)}) \leqslant \tau_j^{(n)} \leqslant t_j^{(n)},$$

for $j = 1, 2,..., n$, and $n = 1, 2,...$. Then

$$\lim_{n\to\infty} R_n(f) = \int_{0+}^{1} f(t)\,dt \tag{3}$$

if f is dominantly integrable. In particular, for such an f,

$$\lim_{n\to\infty} \left(\frac{1}{n} \sum_{j=1}^{n} f\left(\frac{j - (1/2)}{n}\right)\right) = \int_{0+}^{1} f(t)\,dt.$$

By Example 2 below, if $(R_n(f))_{n=1}^{\infty}$ is any sequence of compound rules on $[0, 1]$ not involving $f(0)$, and integrating 1 exactly, we have (3) whenever f is dominantly integrable. Also we shall discuss, for suitable functions g, formulas R_n for which $R_n(f) \to \int_{0+}^{1} f\,dg$, the improper Riemann–Stieltjes integral of $f\,dg$, whenever f is dominantly integrable.

DEFINITION 1. Let g be a fixed complex function on [0, 1], bounded but not constant there, having the property that Riemann–Stieltjes integral $\int_0^1 f\,dg$ exists for every complex function f, Riemann integrable on [0, 1].[1] Let δ, $0 < \delta < 1$, be fixed, and let $d(n)$ map the set of positive integers into itself. For $n = 1, 2,\ldots$, let $c_j^{(n)}, j = 1, 2,\ldots, d(n)$, be given complex numbers, and let $t_j^{(n)}$ ($j = 0, 1,\ldots, d(n)$) and $\tau_j^{(n)}$ ($j = 1, 2,\ldots, d(n)$) be given reals satisfying

$$0 = t_0^{(n)} < t_1^{(n)} < \cdots < t_{d(n)}^{(n)} = 1;$$

and

$$\max(t_{j-1}^{(n)}, \delta t_j^{(n)}) \leqslant \tau_j^{(n)} \leqslant t_j^{(n)}, \qquad j = 1, 2,\ldots, d(n).$$

We also assume the existence of positive constants $B \leqslant 1$ and M such that $\tau_j^{(n)} < B$ implies $|c_j^{(n)}| < M$, for $n = 1, 2,\ldots; j = 1, 2,\ldots, d(n)$. For $n = 1, 2,\ldots$, consider the function Φ_n with domain the set of all complex functions h on (0, 1], defined for every such h by

$$\Phi_n(h) = \sum_{j=1}^{d(n)} c_j^{(n)} h(\tau_j^{(n)})(t_j^{(n)} - t_{j-1}^{(n)}). \tag{3a}$$

We assume, finally, that for every complex function f, Riemann integrable on [0, 1], $\Phi_n(f) \to \int_0^1 f\,dg$. Under the above conditions we shall call $(\Phi_n)_{n=1}^{\infty}$ a "Q-sequence" (Q for "quadrature") or a Q-sequence corresponding to g.

THEOREM 2. *A necessary (sufficient) condition for the existence of a Q-sequence corresponding to a given complex function g as in the first sentence of Definition* 1 *is that g be continuous and of bounded variation on* [0, 1] *(be absolutely continuous on* [0, 1]*), and satisfy a Lipschitz condition on some* $[0, \theta]$, $0 < \theta \leqslant 1$.

THEOREM 3. *A complex function f on* (0, 1] *is dominantly integrable if $\int_{0+}^1 f(x)\,dx$ converges, and, for each Q-sequence $(\Phi_n)_{n=1}^{\infty}$ corresponding to $g(t) = t$, $\Phi_n(f)$ converges to $\int_{0+}^1 f(t)\,dt$. Conversely, if f is dominantly integrable, then, for each g as in the first sentence of Definition* 1, *and, for each Q-sequence $(\Phi_n)_{n=1}^{\infty}$ corresponding to it, $\int_{0+}^1 f(t)\,dg(t)$ converges, and*

$$\Phi_n(f) \to \int_{0+}^1 f(t)\,dg(t).$$

[1] A necessary (sufficient) condition for g to have this property is that g be continuous and of bounded variation on [0, 1] (be absolutely continuous on [0, 1]). Cf. the proof of Theorem 2.

EXAMPLE 2. Suppose $(R_n(f))_{n=1}^{\infty}$ is a sequence of compound rules on $[0, 1]$ not involving $f(0)$, and integrating 1 exactly, namely,

$$R_n(f) = \sum_{k=1}^{n} \sum_{r=1}^{m} w_r n^{-1} f((k-1)\, n^{-1} + x_r n^{-1}), \qquad n = 1, 2,...,$$

where $w_1, ..., w_m$ are given complex constants with

$$\sum_{j=1}^{n} w_j = 1, \qquad \text{and} \qquad 0 < x_1 < \cdots < x_m \leqslant 1.$$

For $n = 2, 3,...,$ arrange the numbers $(k-1)\, n^{-1} + x_r n^{-1}$, $k = 1, 2,..., n$, $r = 1, 2,..., m$ (they are all distinct) as a (strictly) monotone increasing sequence $(\tau_j^{(n)})_{j=1}^{nm}$, and set $t_0^{(n)} = 0$, $t_j^{(n)} = \frac{1}{2}(\tau_j^{(n)} + \tau_{j+1}^{(n)})$ $(j = 1, 2,..., nm - 1)$, $t_{nm}^{(n)} = 1$. Observe that the definition of $(\tau_j^{(n)})_{j=1}^{nm}$ associates with each $n > 1$ and $j = 1, 2,..., nm$ a unique r, $1 \leqslant r \leqslant m$. Given such n and j, use the corresponding r to define

$$c_j^{(n)} = \frac{w_r}{n(t_j^{(n)} - t_{j-1}^{(n)})}.$$

There exists an $M_1^{-1} > 0$ (independent of j and n) such that every $t_j^{(n)} - t_{j-1}^{(n)} > (M_1 n)^{-1}$; thus, each $| c_j^{(n)} | < M$, M being a constant. One may verify that, for $n = 2, 3,..., j = 1, 2,..., nm$,

$$\tau_j^{(n)}/t_j^{(n)} \geqslant \delta = 2[1 + \max_{1 \leqslant r \leqslant m} (x_{r+1}/x_r)]^{-1}, \qquad \text{where} \quad x_{m+1} = 1 + x_1 .$$

It is known (cf. [2, Section 2.4]) that $R_n(f) \to \int_0^1 f(x)\, dx$ for every complex function f, Riemann integrable on $[0, 1]$. Hence, by Theorem 3, $R_n(f)$ converges to $\int_{0+}^1 f(t)\, dt$ for all dominantly integrable functions f.

It often takes some effort to write a quadrature formula as

$$\sum_{j=1}^{d(n)} c_j^{(n)} h(\tau_j^{(n)})(t_j^{(n)} - t_{j-1}^{(n)}),$$

if it was not given in that form originally. This is particularly true of "Gauss-type" quadrature formulas. The next two theorems address this problem.

DEFINITION 2. Assume the first sentence of Definition 1. Let $d(n)$ map the set of positive integers into itself. For $n = 1, 2,...$, let $w_j^{(n)}$ $(j = 1, 2,... d(n))$ be given complex numbers, and let

$$0 = \tau_0^{(n)} < \tau_1^{(n)} < \cdots < \tau_{d(n)}^{(n)} \leqslant 1 = \tau_{d(n)+1}^{(n)}$$

be given reals. We assume the existence of positive constants B^* and M^* such that $\tau_j^{(n)} < B^*$ implies

$$| w_j^{(n)} | < M^* \min(\tau_{j+1}^{(n)} - \tau_{j-1}^{(n)}, \tau_j^{(n)}) \tag{4}$$

for $n = 1, 2,...; j = 1, 2,..., d(n)$. For $n = 1, 2,...$, consider the functional Φ_n^* with domain the set of all complex functions h on $(0, 1]$, defined by

$$\Phi_n^*(h) \equiv \sum_{j=1}^{d(n)} w_j^{(n)} h(\tau_j^{(n)}).$$

We assume, finally, that for every complex function f, Riemann integrable on $[0, 1]$, $\Phi_n^*(f) \to \int_0^1 f\, dg$. Under these conditions we shall call $(\Phi_n^*)_{n=1}^{\infty}$ a "Q^*-sequence" or a Q^*-sequence corresponding to g.

THEOREM 4. *Given a Q-sequence $(\Phi_n)_{n=1}^{\infty}$ corresponding to some g, it is also a Q^*-sequence corresponding to the same g, and conversely.*

The first part of the theorem is immediate: Suppose, first, we never have $\tau_j^{(n)} = \tau_{j+1}^{(n)}$. Set $w_j^{(n)} = c_j^{(n)}(t_j^{(n)} - t_{j-1}^{(n)})$, $n = 1, 2,...; j = 1, 2,..., d(n)$, and observe that, if some $\tau_j^{(n)}$ is $< B$, then (setting $\tau_0^{(n)} = 0$, $\tau_{d(n)+1}^{(n)} = 1$; $n = 1, 2,...$),

$$| w_j^{(n)} | \leqslant | c_j^{(n)} | (\tau_{j+1}^{(n)} - \tau_{j-1}^{(n)}) < M(\tau_{j+1}^{(n)} - \tau_{j-1}^{(n)}),$$

and

$$| w_j^{(n)} | \leqslant | c_j^{(n)} | t_j^{(n)} \leqslant \delta^{-1} | c_j^{(n)} | \tau_j^{(n)} < \delta^{-1} M \tau_j^{(n)};$$

so that

$$| w_j^{(n)} | < \delta^{-1} M \min(\tau_{j+1}^{(n)} - \tau_{j-1}^{(n)}, \tau_j^{(n)}). \tag{4a}$$

Whenever $\tau_j^{(n)} = \tau_{j+1}^{(n)}$ $(n \geqslant 1, 1 \leqslant j < d(n))$, we combine the two summands in (3a) corresponding to j and $j + 1$ into one, thus forming new sequences of τ's and t's, and corresponding sequences of c's and w's. For these new τ's and w's, (4a) holds with M replaced by $2M$.

Putting together Theorem 3 of [9], which gives necessary and sufficient conditions for dominant integrability, and Theorems 3 and 4 above, we immediately obtain

THEOREM 5. (a) *In order, for a complex function f on* $(0, 1]$, *to be such that $\int_{0+}^1 f(t)\, dt$ converges and, for each Q^*-sequence $(\Phi_n^*)_{n=1}^{\infty}$ corresponding to $g(t) \equiv t$, $\Phi_n^*(f) \to \int_{0+}^1 f(t)\, dt$, it is necessary that f be dominantly integrable, i.e., that*

(i) *f be Riemann integrable on each closed subinterval of* $(0, 1]$, *and*

(ii) *there exists a monotone nonincreasing improperly Riemann integrable function h on* $(0, 1]$ *such that, at each point of the interval,* $h(t) \geq |f(t)|$.

(b) *Dominant integrability of a function f is sufficient to guarantee that, given a Q^*-sequence $(\Phi_n{}^*)_{n=1}^{\infty}$ corresponding to some g, $\int_{0+}^{1} f(t)\,dg(t)$ converges and $\Phi_n{}^*(f) \to \int_{0+}^{1} f(t)\,dg(t)$.*

EXAMPLE 3. Given a, b $(-\infty < a < b < \infty)$, dominant integrability on $(a, b]$ of a complex function f will mean dominant integrability of $f(a + t(b - a))$. The concept of a Q^*-sequence carries over, too, to $(a, b]$ with the changes that in Definition 2, 0 is replaced by a, 1 by b, and in its fourth sentence, $\tau_j^{(n)}$ is replaced by $\tau_j^{(n)} - a$ (similarly for the concept of a Q-sequence). The analogs of Theorems 1–5 also hold.

Let $w(t) \equiv (1 - t)^{\alpha} (1 + t)^{\beta}$ where $-\frac{1}{2} \leqslant \alpha \leqslant \frac{1}{2}$, $-\frac{1}{2} \leqslant \beta \leqslant \frac{1}{2}$. We shall next show, as an example of the power of the above theorems, that *for every function f, dominantly integrable on $(-1, 1]$, $\lim_{n\to\infty} Q_n(f/w) = \int_{-1+}^{1} f(t)\,dt$ where, for $n = 1, 2,\ldots$, Q_n is the n-point Gauss–Jacobi quadrature formula corresponding to the weight function w.* For $n = 1, 2,\ldots$, and suitable positive $w_{n,k}$,

$$Q_n(f/w) = \sum_{k=1}^{n} w_{n,k}(w(x_{n,k}))^{-1} f(x_{n,k}),$$

where $x_{n,n} < x_{n,n-1} < \cdots < x_{n,1}$ are the zeros of the Jacobi polynomial $P_n^{(\alpha,\beta)}$. Let $x_{n,0} = 1$, $x_{n,n+1} = -1$ $(n = 1, 2,\ldots)$. Rabinowitz showed (see [10]) that there exist δ_1 in $(0, 1)$ and $c > 0$ such that if, for some n and k $(1 \leqslant k \leqslant n)$, $x_{n,k}$ is in $(1 - \delta_1, 1)$, then

$$w_{n,k}(w(x_{n,k}))^{-1} \leqslant c(x_{n,k-1} - x_{n,k}) \leqslant c \min(x_{n,k-1} - x_{n,k+1}, 1 - x_{n,k}). \quad (5)$$

For $n = 1, 2,\ldots$, consider the functional $Q_n(h/w)$ with domain the set of all complex functions h on $(-1, 1]$. We show that the sequence of these functionals is a Q^*-sequence corresponding to $g(t) \equiv t$. We have in (5) the desired sort of inequality corresponding to (4) for the $w_{n,k}(w(x_{n,k}))^{-1}$ (playing the role of $w_j^{(n)}$), except for the "wrong" endpoint. The weight functions $(1 - t)^{\alpha} \cdot (1 + t)^{\beta}$ are such that, for a given $n = 1, 2,\ldots$, interchanging α and β replaces each $x_{n,k}$ by $-x_{n,n+1-k}$, and each $w_{n,k}$ by $w_{n,n+1-k}$ ([11, (4.1.3), (4.3.3) and (3.4.8)]). Thus $(Q_n(h/w))_{n=1}^{\infty}$ will be shown to be a Q^*-sequence if we can prove that $Q_n(f/w) \to \int_{-1}^{1} f(t)\,dt$ for every f, Riemann integrable on $[-1, 1]$. Let $\alpha(t) = \int_{-1}^{t} w(s)\,ds$ $(-1 \leqslant t \leqslant 1)$, and let $0 < \epsilon < 1$.

For each function f, Riemann integrable on $[-1, 1]$, $\int_{-1+\epsilon}^{1-\epsilon} (w(t))^{-1} f(t)\, d\alpha(t)$ exists and equals $\int_{-1+\epsilon}^{1-\epsilon} f(t)\,dt$ ([7, Theorem 322.1]). Let f_ϵ be f times the characteristic function of $[-1+\epsilon, 1-\epsilon]$. Now $\int_{-1}^{1}(w(t))^{-1}f_\epsilon(t)\,d\alpha(t)$ (where we take $(w(t))^{-1}$ to be, say, 0 at 1 and -1) exists and equals $\int_{-1+\epsilon}^{1-\epsilon} f(t)\,dt$, so by Theorem 15.2.3 of [11], $\lim_{n\to\infty} Q_n(f_\epsilon/w) = \int_{-1+\epsilon}^{1-\epsilon} f(t)\,dt$. Using the first inequality in (5) and its analog for a corresponding interval $(-1, -1+\delta_2)$, we have, if $\epsilon \leqslant \min(\delta_1, \delta_2)$,

$$| Q_n((f-f_\epsilon)/w)| \leqslant 2\tilde{c}\epsilon \sup_{-1\leqslant t\leqslant 1} |f(t)|, \tag{5a}$$

where $\tilde{c}$ is a constant.

It is now easily seen that $\lim_{n\to\infty} Q_n(f/w) = \int_{-1}^{1} f(t)\,dt$. By Theorem 5, $Q_n(f/w) \to \int_{-1+}^{1} f(t)\,dt$ for every f dominantly integrable on $(-1, 1]$.

EXAMPLE 4. We next show that *if* $-\frac{1}{2} \leqslant \alpha \leqslant \frac{1}{2}$, $0 \leqslant \beta \leqslant \frac{1}{2}$, *and if f is dominantly integrable on* $(-1, 1]$, *then the improper Riemann–Stieltjes integral* $\int_{-1+}^{1} f(t)\,d\alpha(t)$ *converges, and equals* $\lim_{n\to\infty} Q_n(f)$.

Using the analog of the first inequality in (5) referred to in the sentence of (5a), and the boundedness of $w(t)$ to the right of but near -1, it follows that, for some constant $c' > 0$, $w_{n,k} \leqslant c'(x_{n,k} - x_{n,k+1}) \leqslant c' \min(x_{n,k-1} - x_{n,k+1}, 1 + x_{n,k})$, if $x_{n,k}$ is in $(-1, -1+\delta_2)$. Thus we only need to know that the Riemann–Stieltjes integral $\int_{-1}^{1} f(t)\,d\alpha(t)$ exists and equals $\lim_{n\to\infty} Q_n(f)$ for all functions f, Riemann integrable on $[-1, 1]$, which is true ([7, Theorem 322.1] and [11, Theorem 15.2.3]).

A special case of Theorem 5 was treated by Miller in [8] (see his Lemma 1 with $T = 1$). He obtained, in effect, for a function f, continuous in $(0, 1]$, (ii) of Theorem 5(a) (with h continuous on $(0, 1]$) a *sufficient* condition that, for each $(\Phi_n{}^*)_{n=1}^{\infty}$ in a certain class of Q^*-sequences corresponding to $g(t) \equiv t$, $\Phi_n{}^*(f) \to \int_{0+}^{1} f(t)\,dt$. Our following lemma allows one to see that, as asserted, Miller's sequences $(\Phi_n{}^*)_{n=1}^{\infty}$ are Q^*-sequences corresponding to $g(t) \equiv t$.

LEMMA 1. *Suppose* $(\Phi_n{}^*)_{n=1}^{\infty}$ *is as in Definition* 2, *except for the following changes: each* $w_j^{(n)} \geqslant 0$, $g(t) \equiv t$, *and we only assume that* $\Phi_n{}^*(f) \to \int_0^1 f(t)\,dt$ *for each real function* f, *continuous in* $[0, 1]$. *Then* $(\Phi_n{}^*)_{n=1}^{\infty}$ *is a* Q^*-*sequence corresponding to* $g(t) \equiv t$.

(Using Lemma 1 in Example 3, it suffices there to show that $Q_n(f/w) \to \int_{-1}^{1} f(t)\,dt$ for all real functions f, continuous in $[-1, 1]$.)

2. PROOF OF THEOREMS 2 AND 4

Proof of Theorem 4. Consider a Q^*-sequence. Let $B = \min(B^*, \frac{1}{2})$ and $M = 2M^*$. For $n = 1, 2, \ldots$, set $t_0^{(n)} = 0$, $t_{d(n)}^{(n)} = 1$; and if $1 \leqslant j < d(n)$, set

$t_j^{(n)} = \frac{1}{2}(\tau_{j+1}^{(n)} + \tau_j^{(n)})$ if $\tau_{j+1}^{(n)} \leqslant 4\tau_j^{(n)}$, and $t_j^{(n)} = 2\tau_j^{(n)}$ if $\tau_{j+1}^{(n)} > 4\tau_j^{(n)}$; also set $c_j^{(n)} = w_j^{(n)}(t_j^{(n)} - t_{j-1}^{(n)})^{-1}$, $j = 1, 2, \ldots, d(n)$. Let $n \geqslant 1$. We shall first show that, if $1 \leqslant j < d(n)$, then $\min\{\tau_j^{(n)}(t_j^{(n)} - t_{j-1}^{(n)})^{-1}, (\tau_{j+1}^{(n)} - \tau_{j-1}^{(n)})(t_j^{(n)} - t_{j-1}^{(n)})^{-1}\} \leqslant 2$. If $1 < j < d(n)$, we have four possibilities for $t_j^{(n)} - t_{j-1}^{(n)}$. The first is $\tau_j^{(n)} \leqslant 4\tau_{j-1}^{(n)}$, and $\tau_{j+1}^{(n)} \leqslant 4\tau_j^{(n)}$. In this case $(\tau_{j+1}^{(n)} - \tau_{j-1}^{(n)})(t_j^{(n)} - t_{j-1}^{(n)})^{-1} = 2$. The second is $4\tau_{j-1}^{(n)} < \tau_j^{(n)} < 4\tau_j^{(n)} < \tau_{j+1}^{(n)}$. In this case $\tau_j^{(n)}(t_j^{(n)} - t_{j-1}^{(n)})^{-1} < \frac{2}{3}$. The third is $\tau_{j+1}^{(n)} > 4\tau_j^{(n)}$, and $4\tau_{j-1}^{(n)} \geqslant \tau_j^{(n)}$, in which case $\tau_j^{(n)}(t_j^{(n)} - t_{j-1}^{(n)})^{-1} < 1$. Finally, the fourth is $4\tau_{j-1}^{(n)} < \tau_j^{(n)} < \tau_{j+1}^{(n)} \leqslant 4\tau_j^{(n)}$, and then $\tau_j^{(n)}(t_j^{(n)} - t_{j-1}^{(n)})^{-1} < 2$.

If $j = 1 < d(n)$, there are two possible cases for $t_j^{(n)} - t_{j-1}^{(n)}$:

(i) $t_1^{(n)} - t_0^{(n)} = t_1^{(n)} = \frac{1}{2}(\tau_2^{(n)} + \tau_1^{(n)})$, and so $\tau_1^{(n)}(t_1^{(n)} - t_0^{(n)})^{-1} < 1$,
and (ii) $t_1^{(n)} - t_0^{(n)} = t_1^{(n)} = 2\tau_1^{(n)}$, in which case $\tau_1^{(n)}(t_1^{(n)} - t_0^{(n)})^{-1} = \frac{1}{2}$.

Thus, if $1 \leqslant j < d(n)$, and $\tau_j^{(n)} < B$, we have $| c_j^{(n)} | < M$. If $\tau_{d(n)}^{(n)} < B \leqslant \frac{1}{2}$, then $| c_{d(n)}^{(n)} | = | w_{d(n)}^{(n)} |(1 - t_{d(n)-1}^{(n)})^{-1} < 2M^* = M$.

There is a constant $\delta_0 \in (0, 1)$ such that $\tau_{d(n)}^{(n)} \geqslant \delta_0$ for $n = 1, 2, \ldots$. For, otherwise, we choose $c \in (0, 1)$, and a subsequence $(k_n)_{n=1}^{\infty}$ of $1, 2, \ldots$ such that $g(c) \neq g(1)$, and $0 < \tau_{d(k_n)}^{(k_n)} < c$ for $n = 1, 2, \ldots$. Let f_c be the characteristic function of $[c, 1]$. Then $0 \equiv \Phi_{k_n}^*(f_c) \to g(1) - g(c) \neq 0$. Since $(2/5)\, t_j^{(n)} \leqslant \tau_j^{(n)}$ if $n \geqslant 1$ and $1 \leqslant j < d(n)$, we can take $\delta = \min(2/5, \delta_0)$. This completes the proof of Theorem 4.

LEMMA 2. *Let $0 \leqslant x \leqslant 1$ and let $(\Phi_n)_{n=1}^{\infty}$ be a Q-sequence corresponding to a function g. We require that g is constant on no closed (nondegenerate) subinterval of* $[0, 1]$ *containing x. For each $\epsilon > 0$ there exists an integer $n(\epsilon) \geqslant 1$ such that if $n \geqslant n(\epsilon)$, $1 \leqslant j \leqslant d(n)$, $t_{j-1}^{(n)} \leqslant x \leqslant t_j^{(n)}$ and $\tau_j^{(n)} < B$, then $| c_j^{(n)} |(t_j^{(n)} - t_{j-1}^{(n)}) < \epsilon$.*

Proof. Suppose Lemma 2 is false. Then there exist $\epsilon > 0$, a subsequence $(N(n))_{n=1}^{\infty}$ of $1, 2, 3, \ldots$, and points p_1, p_2, p_3, $0 \leqslant p_1 \leqslant p_2 \leqslant p_3 \leqslant 1$, $p_1 < p_3$, $p_1 \leqslant x \leqslant p_3$, such that, given any $\epsilon_1 > 0$, if n is sufficiently large, there exists $j = j(n)$, $1 \leqslant j \leqslant d(N(n))$, with $t_{j-1}^{(N(n))} \leqslant x \leqslant t_j^{(N(n))}$, $\tau_j^{(N(n))} < B$,

$$| c_j^{(N(n))} | \, (t_j^{(N(n))} - t_{j-1}^{(N(n))}) \geqslant \epsilon, \qquad \text{and} \qquad \max\{| p_1 - t_{j-1}^{(N(n))} |,$$

$| p_2 - \tau_j^{(N(n))} |, | p_3 - t_j^{(N(n))} |\} < \epsilon_1$. Suppose $p_1 < p_2 < p_3$. Let

$$0 < \epsilon_1 < \tfrac{1}{2}\min\{p_2 - p_1, p_3 - p_2\}. \tag{6}$$

We see, using $(\Phi_n)_{n=1}^{\infty}$, that the characteristic function of every $[a, b] \subseteq [p_1 + \epsilon_1, p_2 - \epsilon_1]$ and of every $[a, b] \subseteq [p_2 + \epsilon_1, p_3 - \epsilon_1]$ has 0 as its Riemann–Stieltjes integral dg on $[0, 1]$. Thus g is constant on (p_1, p_2) and on (p_2, p_3). Since g is continuous on $[0, 1]$, it is constant on $[p_1, p_3]$ which contains x, a contradiction. If $p_1 = p_2$ or $p_2 = p_3$, the argument is essentially the same (but simpler). This proves Lemma 2.

Suppose that g is a complex function, absolutely continuous but non-constant on [0, 1] which satisfies a Lipschitz condition on some $[0, \theta]$, $0 < \theta \leqslant 1$, and is constant on some $[\alpha, \beta]$, $0 < \alpha < \beta < 1$. Then it is not difficult to see that $(\Phi_n)_{n=1}^{\infty}$ is a Q-sequence corresponding to g where, for every complex function h on (0, 1],

$$\begin{aligned}\Phi_n(h) = &\left\{\sum_{j=1}^{2^n} [g(j2^{-n}\alpha) - g((j-1)\,2^{-n}\alpha)]\, h(j2^{-n}\alpha)\right\}\\ &+ \left(\frac{\alpha+\beta}{2} - \alpha\right) h\left(\frac{\alpha+\beta}{2}\right) - \left(\beta - \frac{\alpha+\beta}{2}\right) h\left(\frac{\alpha+\beta}{2}\right)\\ &+ \sum_{j=1}^{2^n} [g(\beta + j2^{-n}(1-\beta))\\ &- g(\beta + (j-1)\,2^{-n}(1-\beta))]\, h(\beta + j2^{-n}(1-\beta)),\end{aligned}$$

$n = 1, 2,\ldots$ (Here, for $n = 1, 2,\ldots$, the numbers $t_0^{(n)}, t_1^{(n)},\ldots, t_{d(n)}^{(n)}$ are, respectively, $0, 1 \cdot 2^{-n}\alpha, 2 \cdot 2^{-n}\alpha,\ldots, 2^n \cdot 2^{-n}\alpha, (\alpha+\beta)/2, \beta, \beta + 1 \cdot 2^{-n}(1-\beta),\ldots, \beta + 2^n \cdot 2^{-n}(1-\beta)$; $\delta = (\alpha+\beta)/(2\beta)$, and $B = \theta$. Thus the case excluded in Lemma 2 can occur.

Proof of Theorem 2. As is well known ([7], Theorem 317) if f is Riemann–Stieltjes integrable dg on [0, 1], f and g cannot have a common point of discontinuity there. Hence g must be continuous in [0, 1]. By [7], Theorem 335, if g is not of bounded variation in [0, 1], then there exists a function, continuous in [0, 1], which is not Riemann–Stieltjes integrable dg on [0, 1].

Again, by [7], Theorem 322.1, if ϕ is absolutely continuous on [0, 1], the Riemann–Stieltjes integral $\int_0^1 f\,d\phi$ exists for each function f, Riemann integrable on [0, 1].

To see the sufficiency of absolute continuity on [0, 1] together with a Lipschitz condition on some $[0, \theta]$, consider any sequence ($n = 1, 2,\ldots$) of sums of the form $\sum_{j=1}^{n} c_j^{(n)} h(t_j^{(n)})(t_j^{(n)} - t_{j-1}^{(n)})$ where, for $n = 1, 2,\ldots, 0 = t_0^{(n)} < t_1^{(n)} < \cdots < t_n^{(n)} = 1$ and each $c_j^{(n)} = [g(t_j^{(n)}) - g(t_{j-1}^{(n)})](t_j^{(n)} - t_{j-1}^{(n)})^{-1}$, and where $\max_{1 \leqslant j \leqslant n} \{t_j^{(n)} - t_{j-1}^{(n)}\} \to 0$. Setting $B = \theta$, we have a Q-sequence corresponding to g.

Now we show necessity of a Lipschitz condition. Let $0 \leqslant \alpha < \beta < B$. We shall show that $|g(\beta) - g(\alpha)| \leqslant M(\beta - \alpha)$. Let $\chi_{\alpha,\beta}$ be the characteristic function of (α, β), and, for $n = 1, 2,\ldots$, let $t_{j_n'-1}^{(n)} \leqslant \alpha \leqslant t_{j_n'}^{(n)}$, $t_{j_n''-1}^{(n)} \leqslant \beta \leqslant t_{j_n''}^{(n)}$. Then

$$|\Phi_n(\chi_{\alpha,\beta})| = \left|\sum_{j=j_n'}^{j_n''} c_j^{(n)} \chi_{\alpha,\beta}(\tau_j^{(n)})(t_j^{(n)} - t_{j-1}^{(n)})\right| < M(\beta - \alpha) + S_n + T_n$$

where

$$S_n = | c_{j'_n}^{(n)} | (t_{j'_n}^{(n)} - t_{j'_n-1}^{(n)})$$

$$\text{if} \quad \tau_{j'_n}^{(n)} \in (\alpha, \beta), \quad S_n = 0 \text{ otherwise};$$

$$T_n = | c_{j''_n}^{(n)} | (t_{j''_n}^{(n)} - t_{j''_n-1}^{(n)})$$

$$\text{if} \quad \tau_{j''_n}^{(n)} \in (\alpha, \beta), \quad T_n = 0 \text{ otherwise}; \quad n = 1, 2, \ldots .$$

We may assume $S_n + T_n \not\to 0$, as otherwise

$$| g(\beta) - g(\alpha)| = \left| \int_0^1 \chi_{\alpha,\beta}\, dg \right| = \lim_{n\to\infty} | \Phi_n(\chi_{\alpha,\beta})| \leqslant M(\beta - \alpha).$$

By Lemma 2, there is a closed (nondegenerate) subinterval of [0, 1] containing α or β, over which g is constant. We may clearly assume $g(\beta) \neq g(\alpha)$.

Let $\alpha_1 = \sup\{x : \alpha \leqslant x < \beta, g(x) = g(\alpha)\}$. Then $\alpha \leqslant \alpha_1 < \beta$, $g(\alpha_1) = g(\alpha)$. Let $\alpha_1 < \alpha_1 + \epsilon' \leqslant \beta$. The set of complex numbers which are constant values of g in closed (nondegenerate) subintervals of $I = (\alpha_1, \alpha_1 + \epsilon')$ is at most denumerable, while the image of I by g is not. Hence, there is an $\alpha_2 \in I$ such that g is constant on no closed (nondegenerate) subinterval of [0, 1] containing α_2.

Similarly, let $\beta_1 = \inf\{x : \alpha_1 < x \leqslant \beta, g(x) = g(\beta)\}$. Then $\alpha_1 < \beta_1 \leqslant \beta$, $g(\beta_1) = g(\beta)$. Let $\alpha_1 \leqslant \beta_1 - \epsilon'' < \beta_1$. Then, analogously, there is $\beta_2 \in (\beta_1 - \epsilon'', \beta_1)$ such that g is constant on no closed (nondegenerate) subinterval of [0, 1] containing β_2.

Let $0 < \epsilon \leqslant (\beta_1 - \alpha_1)/2$, and take $\epsilon' = \epsilon'' = \epsilon$. Then $\alpha_1 < \alpha_2 < \alpha_1 + \epsilon \leqslant \beta_1 - \epsilon < \beta_2 < \beta_1$, and by the above, $| g(\beta_2) - g(\alpha_2)| \leqslant M(\beta_2 - \alpha_2)$. By continuity,

$$| g(\beta) - g(\alpha)| = | g(\beta_1) - g(\alpha_1)| \leqslant M(\beta_1 - \alpha_1) \leqslant M(\beta - \alpha).$$

3. Proof of Theorems 1 and 3

Next we prove Theorem 1. Assume the conditions involving I, δ_1, ϵ, and $m(\epsilon)$. Suppose we knew that (*) $\lim_{t\to 0+} tf(t) = 0$. Given $\epsilon > 0$, let $\chi_1 \in (0, 1)$ be such that $| tf(t)| < \epsilon/2$ whenever $0 < t < \chi_1$, and let $\delta = \min(m(\epsilon/2), 1 - \delta_1)$, $\chi = \min(\chi_1, m(\epsilon/2))$. Suppose $0 < t_0 < t_1 < \cdots < t_n = 1$, $t_0 < \chi$; $t_{j-1} \leqslant \tau_j \leqslant t_j$, and $t_{j-1}t_j^{-1} > 1 - \delta$, $j = 1, 2, \ldots, n$. Set $\tau_0 = t_0$, $t_{-1} = 0$. Then $| I - \sum_{j=1}^{n} f(\tau_j)(t_j - t_{j-1})| = | I + t_0 f(t_0) - \sum_{j=0}^{n} f(\tau_j)(t_j - t_{j-1})| <$

$|I - \sum_{j=0}^{n} f(\tau_j)(t_j - t_{j-1})| + (\epsilon/2) < (\epsilon/2) + (\epsilon/2) = \epsilon$. Hence ([9], Definition 1) f is dominantly integrable, and ([9], Theorem 1) $I = \int_{0+}^{1} f(t)\,dt$.

Suppose (*) is false. Choose F, one of the functions $\pm\mathrm{Re} f$, $\pm\mathrm{Im} f$, so that, for some positive ϵ and some sequence $(s_j)_{j=1}^{\infty}$ with $\frac{1}{2} > s_1 > \frac{1}{2}s_1 > s_2 > \frac{1}{2}s_2 > \cdots > 0$, each $s_j F(s_j) \geqslant \epsilon$. Choose an integer $N_1 \geqslant 1$ such that $2^{-N_1} \leqslant m(1)$. For $N = N_1 + 1, N_1 + 2, \ldots,$ set $t_0^{(N)} = 0$; $t_j^{(N)} = s_{N-j}$, $j = 1, 2, \ldots, N - N_1$; and $t_j^{(N)} = s_{N_1} + (j - N + N_1)\, 2^{-N_1}(1 - s_{N_1})$, $j = N - N_1 + 1, N - N_1 + 2, \ldots, N - N_1 + 2^{N_1} = j(N, N_1)$. Then, if $N > N_1$,

$$\left[\sum_{j=1}^{j(N,N_1)} F(t_j^{(N)})(t_j^{(N)} - t_{j-1}^{(N)})\right] - \sum_{j=N-N_1+1}^{j(N,N_1)} F(t_j^{(N)})(t_j^{(N)} - t_{j-1}^{(N)})$$

$$= \sum_{j=1}^{N-N_1} F(t_j^{(N)})(t_j^{(N)} - t_{j-1}^{(N)}) > \tfrac{1}{2} \sum_{j=1}^{N-N_1} F(t_j^{(N)})\, t_j^{(N)} \geqslant \tfrac{1}{2}(N - N_1)\,\epsilon,$$

and, hence,

$$\sum_{j=1}^{j(N,N_1)} F(t_j^{(N)})(t_j^{(N)} - t_{j-1}^{(N)})$$

$$> 2^{-N_1}(1 - s_{N_1}) \left[\sum_{j=1}^{2^{N_1}} F(s_{N_1} + (2^{N_1} + 1 - j)\, 2^{-N_1}(1 - s_{N_1}))\right] + \tfrac{1}{2}(N - N_1)\,\epsilon$$

which is arbitrarily large if N is sufficiently large. This contradicts the fact that, for every $N > N_1$,

$$\left| I - \sum_{j=1}^{j(N,N_1)} f(t_j^{(N)})(t_j^{(N)} - t_{j-1}^{(N)}) \right| < 1.$$

Hence, (*) holds.

Now we show the nonparenthetical "only if" part of Theorem 1. By Theorem 3 of [9], f is Riemann integrable on each $[a, b] \subset (0, 1]$, and there exists a function h, monotone nonincreasing and improperly Riemann integrable on $(0, 1]$ such that $h(t) \geqslant |f(t)|$ throughout $(0, 1]$.

Taking an $\epsilon_1 \in (0, 1)$ with $\int_{0+}^{\epsilon_1} h(\delta t)\,dt < \epsilon/4$, we may write the sum occurring in (2), assuming (1), if each $t_j - t_{j-1}$ is sufficiently small, as

$$\sum_{j=1}^{n_1-1} f(\tau_j)(t_j - t_{j-1}) + \sum_{j=n_1}^{n} f(\tau_j)(t_j - t_{j-1})$$

where $t_{n_1-1} \leqslant \epsilon_1 < t_{n_1}$, $1 < n_1 < n$.

Now

$$\left|\left\{\sum_{j=n_1}^{n} f(\tau_j)(t_j - t_{j-1})\right\} - \int_{t_{n_1-1}}^{1} f(t)\,dt\right|$$

$$\leqslant \left|\left\{\sum_{j=n_1+1}^{n} f(\tau_j)(t_j - t_{j-1})\right\} + f(\epsilon_1)(t_{n_1} - \epsilon_1) - \int_{\epsilon_1}^{1} f(t)\,dt\right|$$

$$+ [h(\delta\epsilon_1) + h(\epsilon_1)](t_{n_1} - t_{n_1-1}) + \int_{0+}^{\epsilon_1} h(t)\,dt < \epsilon/2,$$

if each $t_j - t_{j-1}$ is sufficiently small. So, then,

$$\left|\sum_{j=1}^{n} f(\tau_j)(t_j - t_{j-1}) - \int_{0+}^{1} f(t)\,dt\right|$$

$$< (\epsilon/2) + \left|\sum_{j=1}^{n_1-1} f(\tau_j)(t_j - t_{j-1})\right| + \int_{0+}^{\epsilon_1} |f(t)|\,dt$$

$$\leqslant (\epsilon/2) + \sum_{j=1}^{n_1-1} h(\delta t_j)(t_j - t_{j-1}) + \int_{0+}^{\epsilon_1} h(t)\,dt$$

$$\leqslant (\epsilon/2) + 2\int_{0+}^{\epsilon_1} h(\delta t)\,dt < \epsilon.$$

This completes the proof of Theorem 1.

LEMMA 3. *If f is Riemann integrable on each $[a, b] \subset (0, 1]$, and $|f|$ is improperly Riemann integrable on $(0, 1]$, then $\int_{0+}^{1} f\,dg$ converges for all g as in the first sentence of Definition 1 for which a Q-sequence corresponding to it exists.*

Proof. By Definition 1, the Riemann–Stieltjes integral $\int_{\epsilon_1}^{1} f\,dg$ exists for every ϵ_1, $0 < \epsilon_1 \leqslant 1$. We show that, given any $\epsilon > 0$, if $\epsilon_1 > 0$ is sufficiently small, $|\int_{\epsilon_2}^{\epsilon_1} f\,dg| < \epsilon$ for all ϵ_2, $0 < \epsilon_2 < \epsilon_1$. Now (see Theorem 2), if $0 < \epsilon_2 < \epsilon_1 \leqslant \theta$, we have for every partition $\epsilon_2 = t_0 < t_1 < \cdots < t_n = \epsilon_1$,

$$\left|\sum_{j=1}^{n} f(t_j)(g(t_j) - g(t_{j-1}))\right| \leqslant M \sum_{j=1}^{n} |f(t_j)|\,(t_j - t_{j-1}),$$

M being a constant $\geqslant 0$.

Thus

$$\left|\int_{\epsilon_2}^{\epsilon_1} f\,dg\right| \leqslant M \int_{\epsilon_2}^{\epsilon_1} |f|\,dt \leqslant M \int_{0+}^{\epsilon_1} |f|\,dt \to 0, \qquad \text{as} \quad \epsilon_1 \to 0+.$$

This proves Lemma 3.

We next prove Theorem 3. By Theorem 1, if f is not dominantly integrable, then, for every δ, $0 < \delta < 1$, there is a Q-sequence $(\Phi_n)_{n=1}^{\infty}$ corresponding to $g(t) \equiv t$ (each $\Phi_n(h)$ being a Riemann sum) such that $(\Phi_n(f))_{n=1}^{\infty}$ diverges. We prove now the statement following "Conversely" in Theorem 3.

Consider a g as in the first sentence of Definition 1, and a Q-sequence $(\Phi_n)_{n=1}^{\infty}$ corresponding to it:

$$\Phi_n(h) = \sum_{j=1}^{d(n)} c_j^{(n)} h(\tau_j^{(n)})(t_j^{(n)} - t_{j-1}^{(n)}).$$

Let $(\eta_k)_{k=1}^{\infty}$ denote a (strictly) decreasing sequence with $\eta_1 < B$ and limit 0. For $n, k = 1, 2, 3,\ldots$ define $j(n, k)$ by $t_{j(n,k)-1}^{(n)} \leqslant \eta_k < t_{j(n,k)}^{(n)}$. By Theorems 3 and 1 of [9], and by Lemma 3 above, $\int_{0+}^{1} f\,dg$ converges. For $n, k = 1, 2,\ldots$, with $\chi_{\delta\eta_k,1}$ denoting the characteristic function of $[\delta\eta_k, 1]$, we have:

$$\begin{aligned}
&\Big| \int_{0+}^{1} f\,dg - \sum_{j=1}^{d(n)} c_j^{(n)} f(\tau_j^{(n)})(t_j^{(n)} - t_{j-1}^{(n)}) \Big| \\
&\quad\leqslant \Big| \int_{0+}^{\delta\eta_k} f\,dg \Big| + \Big| \int_{\delta\eta_k}^{1} f\,dg - \sum_{j=1}^{d(n)} c_j^{(n)} \chi_{\delta\eta_k,1}(\tau_j^{(n)}) f(\tau_j^{(n)})(t_j^{(n)} - t_{j-1}^{(n)}) \Big| \\
&\qquad + \sum_{j=1}^{j(n,k)-1} | c_j^{(n)} f(\tau_j^{(n)})(t_j^{(n)} - t_{j-1}^{(n)})|,
\end{aligned}$$

where $\sum_{j=1}^{0} = 0$.

Let $\epsilon > 0$. If k is sufficiently large, and $n \geqslant 1$, then

$$\Big| \int_{0+}^{\delta\eta_k} f\,dg \Big| < \epsilon/3,$$

and

$$\begin{aligned}
\sum_{j=1}^{j(n,k)-1} | c_j^{(n)} f(\tau_j^{(n)})(t_j^{(n)} - t_{j-1}^{(n)})| &\leqslant M \sum_{j=1}^{j(n,k)-1} \hat{f}(\delta t_j^{(n)})(t_j^{(n)} - t_{j-1}^{(n)}) \\
&\leqslant M \int_{0+}^{\eta_k} \hat{f}(\delta t)\,dt < \epsilon/3,
\end{aligned}$$

where $\hat{f}(t) = \sup_{t \leqslant x \leqslant 1} |f(x)|$, $0 < t \leqslant 1$ (see Corollary 2 of [9], and the first sentence of its proof). For each fixed $k \geqslant 1$, if n is sufficiently large,

$$\Big| \int_{\delta\eta_k}^{1} f\,dg - \sum_{j=1}^{d(n)} c_j^{(n)} \chi_{\delta\eta_k,1}(\tau_j^{(n)}) f(\tau_j^{(n)})(t_j^{(n)} - t_{j-1}^{(n)}) \Big| < \epsilon/3,$$

from Definition 1. Theorem 3 is now established.

We conclude with a proof of Lemma 1. Let f be a real function, Riemann integrable on $[0, 1]$. For each positive integer m, let G_m and H_m be step functions on $[0, 1]$ satisfying there $G_m \geqslant f \geqslant H_m$, and such that $\int_0^1 G_m\,dt - (1/m) < \int_0^1 f\,dt < \int_0^1 H_m\,dt + (1/m)$. For $m = 1, 2,\ldots$, there exist real functions g_m and h_m, each continuous on $[0, 1]$, satisfying there $g_m \geqslant G_m$, $h_m \leqslant H_m$, $\int_0^1 g_m\,dt < \int_0^1 G_m\,dt + (1/m)$, and $\int_0^1 h_m\,dt > \int_0^1 H_m\,dt - (1/m)$.

Thus, for $m = 1, 2,\ldots$,

$$\int_0^1 f\,dt + (2/m) > \int_0^1 g_m\,dt = \lim_{n\to\infty} \Phi_n^*(g_m) \geqslant \overline{\lim_{n\to\infty}}\, \Phi_n^*(f) \geqslant \underline{\lim_{n\to\infty}}\, \Phi_n^*(f)$$

$$\geqslant \lim_{n\to\infty} \Phi_n^*(h_m) = \int_0^1 h_m\,dt > \int_0^1 f\,dt - (2/m).$$

Hence $\lim_{n\to\infty} \Phi_n^*(f) = \int_0^1 f\,dt$. This clearly implies that $\Phi_n^*(f) \to \int_0^1 f\,dt$ whenever f is a complex function, Riemann integrable on $[0, 1]$.

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